NUMERICAL SOLUTION OF A PROBLEM IN FORCED CONVECTION

by

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1. INTRODUCTION

The solution of a partial differential equation, with given boundary and initial conditions, for a function θ of two independent variables ζ , η can be sought in the form

$$\theta = \sum_{n=1}^{\infty} f_n (\zeta) \varphi_n (\eta), \qquad (1)$$

where the $f_n (\zeta)$ are definite functions, suitably chosen, and the $\varphi_n (\eta)$ are functions to be found. If the f_n form a 'complete' set there will be an infinite set of ordinary differential equations for the determination of the φ_n ; and if the f_n are sensibly chosen in relation to the problem the early terms of the series (1) may be dominant.

I shall illustrate the method on a specific problem,in which the differential equation is parabolic. The details are of course special but they will have analogues in other problems, so that general morals may be drawn.

The equation to be solved is

$$\frac{\partial \theta}{\partial t} = b \left\{ \frac{\partial^2 \theta}{\partial z^2} - z g^3 (t) \theta \right\} + c g^6(t) \frac{\mathrm{Ai}\left\{ z g (t) - \lambda(t) \right\}}{\mathrm{Ai}\left\{ - \lambda(t) \right\}} \qquad (2)$$

where $\mathrm{Ai}(x)$ is the Airy function, defined and graphed in § 4,

b, c are positive constants,

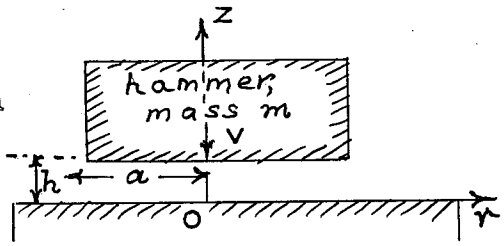
$g(t)$, $\lambda(t)$ are functions whose specification of course affects the arithmetic when we ultimately come to it but does not affect the rather extensive preliminary analysis.

I shall give the physical context, and indicate how (2) is arrived at as on the one hand a reasonable physical approximation and on the other hand an equation that may yield to mathematical attack. Knowledge of this context is of service for suggesting 'ideas' and not merely of interest. The attack on (2) starts at equation (8) in § 3, to which the reader may skip if he pleases.

2. THE PHYSICAL PROBLEM

A thin film of viscous liquid (nitroglycerine, in the case that was of interest) lies on a horizontal surface or 'anvil', and a hammer with circular horizontal under-surface is dropped on to it.

The liquid is thus squeezed from between the anvil and the descending hammer, and since the film is thin its mean radial velocity under the edge of the hammer will (by conservation of volume) be large compared with the velocity V of the hammer. Since moreover the radial velocity is zero on the axis of symmetry the liquid must have a large radial acceleration, so there is a large pressure gradient and the central pressure will be large compared with the (atmospheric) edge-pressure. Hence there is a large upward force on the hammer bringing it (and the liquid) progressively to rest.



Since the liquid is viscous its radial velocity is zero at the bounding surfaces, so the motion is shearing, with dissipation of mechanical energy into heat. This heat is to some extent conveyed away by conduction into the material of the hammer and anvil, and conduction and convection in the liquid.

The complete problem is to calculate (I) the motion of the liquid and the related motion of the hammer, and (II) the temperature of the liquid, as functions of position and time. This problem is complicated. The natural assumptions to make, to simplify it for a first attack, are (i) that the viscosity μ of the liquid is constant, (ii) that the surfaces of the hammer and anvil remain plane and parallel, i.e. are not deformed under the large local pressure-forces applied to them, and (iii) that inertial forces in the liquid are negligible compared to the pressure forces. (In fact, as the table below indicates, in a practical case these assumptions may be seriously in error). Then Part I - calculation of the motion - becomes self-contained and relatively easy, and Part II - calculation of the temperature - becomes a separate problem for which the answers to I provide data.

It is on these assumptions that I proceed, and it is the temperature problem that is here to be considered. The data, drawn from the solution of I, are

$$\text{downward vel. of hammer} = -\frac{dh}{dt} = V = \frac{3\pi\mu a^4}{4m}\left(\frac{1}{h_o^2}-\frac{1}{h^2}\right) + V_o \quad * \qquad (3)$$

$$\text{radial vel. of liquid} = Ur = \frac{3Vr}{h}\cdot\frac{z}{h}\left(1-\frac{z}{h}\right) \qquad (4)$$

$$\text{downward vel. of liquid} = -w = V\left(\frac{3z^2}{h^2}-\frac{2z^3}{h^3}\right)$$

$$\text{central pressure in liquid} = p = \frac{3\mu a^2 V}{h^3}$$

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* From this the dependence of h on t could be calculated, but it is not required.

rate of generation of heat $= D = \frac{\mu}{J}\left\{ r^2 \left(\frac{\partial U}{\partial z}\right)^2 + 12\ U^2 \right\}$, (5)

and if the conductivity K of the liquid were zero its temperature θ at r = a, z = 0 would be $(2.15p/J\ \rho\ s)\ ^{o}C$ [1].

Specimen values, for a = 1, m = 400, $h_c = 5 \times 10^{-2}$, $\mu = 0.25$, $\rho = 1.6$, s = 0.425 (in c.g.s. units):

V_o (cm/sec)	V = 0 for h (cm) =	$p_{max.}$ (atmos.)	θ (^{o}C)
200	2.7×10^{-3}	1400	106
600	1.5×10^{-3}	22 000	1600

3. THE CONDUCTION-CONVECTION EQUATION

is $\frac{\partial \theta}{\partial t} = \frac{K}{\rho s} \nabla^2 \theta - \left(u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} + w \frac{\partial \theta}{\partial z}\right) + \frac{D}{\rho s}$.

It is taken with the initial condition $\theta = 0$ at t = 0 or $h = \infty$ (which from (3) and the preceding table is a reasonable physical approximation), and the boundary condition $\theta = 0$ at z = 0, h; this corresponds to an assumed perfect conductivity in the material of the hammer and anvil - an assumption whose physical acceptability may be shown a posteriori.

Because of the simple form in which u, v, w, D depend on r (see (4) and (5)) this equation is satisfied by

$$\theta = \theta_o + r^2 \theta_1 ,$$

where θ_o, θ_1 depend on z, t only, provided

$$\frac{\partial \theta_1}{\partial t} = \frac{K}{\rho s} \frac{\partial^2 \theta_1}{\partial z^2} - 2U\theta_1 - w \frac{\partial \theta_1}{\partial z} + \frac{\mu}{J \rho s} \left(\frac{\partial U}{\partial z}\right)^2 , \qquad (6)$$

$$\frac{\partial \theta_o}{\partial t} = \frac{K}{\rho s} \frac{\partial^2 \theta_o}{\partial z^2} - w \frac{\partial \theta_o}{\partial z} + \frac{4K}{\rho s} \theta_1 + \frac{12 \mu}{J \rho s} U^2 ,$$

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[1] ρ is the density and s the specific heat of the liquid, and J is Joule's constant.

with θ_0, θ_1 each satisfying the same initial and boundary conditions as θ. In a practical case the contribution of θ_0 is negligible, and the essential problem is to calculate θ_1. The difficulty of this derives from the variation of U, w with t, z, e.g. from (4)

$$U = \frac{3Vz}{h^2}\left(1 - \frac{x}{h}\right), \quad \frac{\partial U}{\partial z} = \frac{3V}{h^2}\left(1 - \frac{2z}{h}\right), \tag{7}$$

where V, h depend on t. We can however make an approximation which is both physically and mathematically acceptable by taking w = 0 and restricting z to be small, so that U is taken as $3Vz/h^2$: a form in which the variables t (via V/h^2) and z are 'separated'. Along with this approximation it is appropriate mathematically to take as the boundary conditions $\theta = 0$ for $z = 0, +\infty$. It is then appropriate to replace the form (7) for $\partial U/\partial z$ by a form which fits (7) near z = 0 but diminishes to 0 as z increases to ∞. Calling this at present

$$\frac{\partial U}{\partial z} \simeq \frac{3V}{h^2} f, \tag{7a}$$

(6) is replaced by

$$\frac{\partial \theta_1}{\partial t} = \frac{K}{\rho s}\frac{\partial^2 \theta_1}{\partial z^2} - \frac{6Vz}{h^2}\theta_1 + \frac{\mu}{J\rho s}\cdot\frac{9V^2}{h^4} f^2 \tag{8}$$

We arrive thus at the form (2), with $g^3(t) = 6\rho sV/Kh^2$ and a special choice for f^2 whose provenance will be seen below.

The coefficient of θ_1 is further simplified if we take instead of z a variable

$$\zeta = z\left(\frac{6\rho s}{K}\cdot\frac{V}{h^2}\right)^{\frac{1}{3}} = z\, g(t), \tag{9}$$

and replace t by the related variable

$$\eta = \frac{1}{100h}; \tag{10}$$

the factor 100 is chosen so that in a practical case η is comparable with 1. After abbreviating by writing

$$W = W(\eta) = V\eta^2 \tag{11}$$

this gives

$$\frac{\partial \theta_1}{\partial \eta} + \frac{1}{3W}\frac{dW}{d\eta}\zeta\frac{\partial \theta_1}{\partial \zeta} = \frac{a}{W^{\frac{1}{3}}}\left(\frac{\partial^2 \theta_1}{\partial \zeta^2} - \zeta\theta_1\right) + kWf^2, \tag{12}$$

where $\alpha = \left(\frac{3600\,K}{\rho s} \right)^{\frac{1}{3}} = 1.55$

$$k = \frac{9 \times 10^{6} \mu}{J \rho s} = 0.079,$$

the numerical values being those for nitroglycerine at ordinary temperatures. The initial and boundary conditions are

$$\theta_1 = 0 \quad \text{for } \eta = 0 \quad \text{and for } \zeta = 0, +\infty .$$

4. THE EIGENFUNCTIONS

If W were constant the homogeneous form of (12) would be satisfied by $\theta_1 = e^{-c\eta} L(\zeta)$ provided

$$L''(\zeta) = \left(\zeta - \frac{c W^{\frac{1}{3}}}{\alpha} \right) L(\zeta) .$$

Of the two fundamental solutions of this, the one which vanishes for $\zeta = +\infty$ is the Airy function [1] $Ai(\zeta - cW^{\frac{1}{3}}/\alpha)$, and to make $\theta_1 = 0$ for $\zeta = 0, -cW^{\frac{1}{3}}/\alpha$ must be a zero of Ai(x). This suggests that as eigenfunctions we use the set

$$L_n(\zeta) = Ai(\zeta - \lambda_n), \tag{13}$$

where $-\lambda_1 , -\lambda_2 , \ldots$ are the zeros of Ai(x), so that

$$L_n''(\zeta) = (\zeta - \lambda_n) L_n(\zeta) ; \tag{14}$$

and we seek a solution of (12) in the form

$$\theta_1 = \sum_{n=1}^{\infty} \varphi_n(\eta) L_n(\zeta). \tag{15}$$

This will give $\theta_1 = 0$ for $\zeta = 0, \infty$ provided the series converges uniformly for $0 \leq \zeta < \infty$ *.

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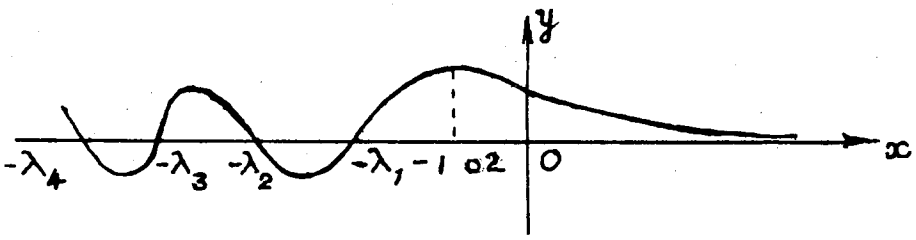
* which is the case, since (22) and (26) give $\varphi_n = 0(\lambda_n^{-\frac{9}{4}}) = 0(n^{-\frac{3}{2}})$.

[1] See H. and B. Jeffreys, Methods of Mathematical Physics, or for a fuller discussion, with tables, British Association Tables Part Volume B (by J.C.P. Miller), 1946.



$y = Ai(x)$

$$\lambda_1 = 2.34,\ \lambda_2 = 4.09,\ \lambda_3 = 5.52,\ \ldots,\ \lambda_\eta \sim \left\{ \frac{3\pi}{2}\left(n + \frac{3}{4}\right) \right\}^{\frac{2}{3}},$$

$$|L_x(0)| \sim 1.59\ \lambda_n^{\frac{1}{4}}$$

For subsequent purposes we need a stock of expansions in the $L_n(\zeta)$, notably

$$\frac{Ai(\zeta - \lambda)}{Ai(-\lambda)} = \sum_{n=1}^{\infty} \frac{L_n(\zeta)}{(\lambda_n - \lambda) L_n'(o)} \quad \text{if } \zeta > 0, \tag{16}$$

$$\zeta Ai'(\zeta - \lambda_m) = \zeta L_m'(\zeta) = -\tfrac{1}{2} L_m(\zeta) + \sum_{n \neq m}{}' \frac{6 L_m'(o) L_n(\zeta)}{(\lambda_n - \lambda_m)^3 L_n'(o)}$$

$$\text{if } \zeta > 0, \tag{17}$$

$$\sum_{n \neq m}{}' \frac{1}{(\lambda_n - \lambda_m)^3} = \tfrac{1}{4} \tag{18}$$

$$\sum_{n \neq m}{}' \frac{1}{(\lambda_m - \lambda_n)^3 (\lambda_n - \lambda)} = \frac{1}{12(\lambda_m - \lambda)} + \frac{\lambda}{3(\lambda_m - \lambda)^2}$$

$$+ \frac{Ai'(-\lambda)}{(\lambda_m - \lambda)^3 Ai(-\lambda)} - \frac{1}{(\lambda_m - \lambda)^4}\ ; \tag{19}$$

and $$\sum_{n \neq m}{}' \frac{e^{-\lambda_n p}}{(\lambda_m - \lambda_n)^3 (\lambda_n - \lambda)} =$$

$$\frac{e^{-\lambda_m p}}{\lambda_m - \lambda} \left(\frac{1}{12} + \frac{\lambda_m p}{3} \frac{\lambda}{3(\lambda_m - \lambda)^2} - \frac{p^3}{6} - \frac{p^2}{2(\lambda_m - \lambda)} - \frac{p}{(\lambda_m - \lambda)^2} - \frac{1}{(\lambda_m - \lambda)^3} \right.$$

$$\left. + O(\lambda_m^{-3}) \right), \tag{20}$$

for λ_m large, provided $p > 0$ and $\lambda < \lambda_1$. These may be proved by countour integration.

Formula (16) suggests a suitable choice for the function f^2, viz

$$f^2 = \frac{Ai(\zeta - \lambda)}{Ai(-\lambda)},$$

which (compare (7) and (7a)) is to approximate to $1 - 4z/h$ for z small, i.e. by (9) and (10)

$$f^2 \approx 1 - 4\zeta\left(\frac{100K}{6\rho s}\right)^{\frac{1}{3}} \left(\frac{\eta}{V}\right)^{\frac{1}{3}} \qquad \text{for } \zeta \text{ small.}$$

The point is that (16) gives a function with the right sort of graph and with a convenient expansion. For $\zeta = 0$ it has the value 1, as desired, and it will have the correct gradient at $\zeta = 0$ if we choose $\lambda = \lambda(\eta)$ so that

$$\frac{Ai'(-\lambda)}{Ai(-\lambda)} = -4\left(\frac{100K}{6\rho s}\right)^{\frac{1}{3}} \left(\frac{\eta}{V}\right)^{\frac{1}{3}}. \qquad (21)$$

For $\eta = 0$ this gives $\lambda = 1.02$, and λ decreases as η increases; but the precise form of $\lambda(\eta)$ is of no consequence until the arithmetic starts.

This choice of f^2 has already been shown in (2).

On substituting the series (15) into (12) and using (14) and the expansions (16), (17) we obtain

$$\frac{d\varphi_n}{d\eta} + \frac{1}{3W}\frac{dW}{d\eta}\left\{-\tfrac{1}{2}\varphi_n + \sum_{m\neq n} \frac{6L_m'(o)\varphi_m}{(\lambda_n - \lambda_m)^3 L_n'(o)}\right\} = -\frac{a\lambda_\eta \varphi_n}{W^{\frac{1}{3}}}$$

$$+ \frac{kW}{(\lambda_n - \lambda)L_n'(o)}$$

This infinite set of differential equations is a little simplified by writing

$$\Psi_n(\eta) = W^{-\frac{1}{6}} L_n'(o)\,\varphi_n(\eta), \qquad (22)$$

which gives

$$\frac{d\Psi_n}{d\eta} + \sum_{m\neq n} \frac{2}{W}\frac{dW}{d\eta}\frac{\Psi_m}{(\lambda_n - \lambda_m)^3} = -\frac{a\lambda_n \Psi_n}{W^{\frac{1}{3}}}$$

$$+ \frac{kW^{\frac{5}{6}}}{\lambda_n - \lambda(\eta)}. \qquad (23)$$

The associated initial conditions are $\Psi_n = 0$ for $\eta = 0$.

The run of the denominators $(\lambda_n - \lambda_m)^3$ is shown by the table

n \ m	1	2	3	4	5
1	.	-5.36	-32.3	-88.0	-176.2
2	5.36		- 2.95	-19.7	- 57.3
3	32.3	2.95		- 2.02	- 14.2
4	88.0	19.7	2.02		- 1.56
5	176.2	57.3	14.2	1.56	

so in (23) we may expect the near-diagonal terms to be dominant.

5. ATTACK ON THE EQUATIONS (23)

We can form some notion of the behaviour of the Ψ_n by neglecting the off-diagonal summation on the left of (23), so that each equation becomes self-contained. Instead of writing the formal 'solution' so obtained it is more instructive to consider approximations to it for η small and for η not small.

(i) η small

For $\eta = 0$, $V = V_o > 0$, so for η small $W = V\eta^2 \approx V_o \eta^2$.

If we omit the first term on the right of (23) as well as the summation on the left we obtain

$$\Psi_n \approx \frac{3}{8} k V_o^{\frac{5}{6}} \frac{\eta^{\frac{8}{3}}}{\lambda_n - \lambda(o)}, \qquad (24)$$

and this approximation gives the term neglected on the right the value

$$\frac{a\lambda_n \Psi_n}{W^{\frac{1}{3}}} = \frac{3}{8} a k V_o^{\frac{1}{2}} \frac{\lambda_n \eta^2}{\lambda_n - \lambda(o)}. \qquad (24a)$$

This is $O(\eta^2)$, whereas the term that was retained on the right is $O(\eta^{\frac{5}{3}})$. This suggests that when η is small $\Psi_n \sim c_n \eta^{\frac{8}{3}}$, but from the form of the residual (24a) the range in which this approximation is 'good' should diminish to zero as n (and hence λ_n) becomes large.

To test whether (24) is really a valid approximation we substitute it in the off-diagonal terms on the left of (23). The sum may be evaluated by means of (19), and we obtain the residual

$$\sum_{m \neq n} \frac{2}{W} \frac{dW}{d\eta} \frac{\Psi_n}{(\lambda_n - \lambda_m)^3} = \frac{1}{8} \frac{k W^{\frac{5}{6}}}{\lambda_n - \lambda(o)} + O\left(\frac{1}{\lambda_n^2}\right), \qquad (24b)$$

which is of the same order as the last term on the right of (23). It is seen immediately that (24) is in error, but only in the numerical coefficient, and the approximation valid for n large and η small is

$$\Psi_n \approx \frac{1}{3} k V_o^{\frac{5}{6}} \frac{\eta^{\frac{8}{3}}}{\lambda_n - \lambda(o)} \quad . \tag{25}$$

(The evaluation of the residual at (24b) is not strictly correct because clearly we cannot assert (25) when n is small; but we may expect (25) to be of the correct order of magnitude when n is small, and the situation is saved since the Ψ's of low order in the summation have coefficients of order λ_n^{-3}).

(ii) η not small

If we neglect both terms on the left of (23) we get the approximation

$$\Psi_n \approx \frac{k W^{\frac{7}{6}}}{\alpha \lambda_n (\lambda_n - \lambda(\eta))} \quad . \tag{26}$$

Since W is regular when η is not small this gives $d\Psi_n/d\eta = O(\lambda_n^{-2})$, while the terms on the right of (23) are separately $O(\lambda_n^{-1})$. Moreover by use of (19), (26) gives

$$\sum_{m \neq n} \frac{2}{W} \frac{dW}{d\eta} \frac{\Psi_n}{(\lambda_n - \lambda_m)^3} = \frac{2k}{\alpha} W^{\frac{1}{6}} \frac{dW}{d\eta} \cdot \frac{1}{\lambda_n (\lambda_n - \lambda)}$$

$$+ O\left(\frac{1}{\lambda_n^3}\right) = O\left(\frac{1}{\lambda_n^2}\right) .$$

Hence (26) is a valid approximation when n is large.

(iii) η unrestricted

The approximations (25), (26) are woolly in that it is not clear how the one merges into the other as η increases. There is however an approximation that includes both and bridges the gap, namely

$$\Psi_n(\eta) \approx k \int_o^{\eta} \frac{W^{-\frac{1}{6}}(\eta) \, W(\eta') d\eta'}{\lambda_n - \lambda(\eta')}$$

$$\times \exp\left\{ -\alpha \lambda_n \int_{\eta'}^{\eta} W^{\frac{2}{3}}(\eta) \, W^{-1}(\eta'') \, d\eta'' \right\} . \tag{27}$$

This formula is to be compared with

$$\Psi_n(\eta) \approx k \int_o^{\eta} \frac{W^{\frac{5}{6}}(\eta') d\eta'}{\lambda_n - \lambda(\eta')} \exp\left\{ -\alpha \lambda_n \int_{\eta'}^{\eta} W^{-\frac{1}{3}}(\eta'') \, d\eta'' \right\} ,$$

which would solve (23) exactly if the off-diagonal terms were absent, and it was obtained by experiment based on the latter. It may be verified either with the help of (20), or by showing that its principal part agrees with (25) or (26) according as η is or is not small; this involves somewhat close argument.

Formula (27) can be regarded as the first term of an asymptotic expansion for Ψ_n when n is large, and it is practicable to obtain further terms; the second approximation is

$$\Psi_n(\eta) \approx k\int_0^{\eta}\left\{ W^{-\frac{1}{6}}(\eta)\ W(\eta')\left(\frac{1}{\lambda_n - \lambda(\eta')} - \frac{\lambda(\eta')}{(\lambda_n - \lambda(\eta'))^2}\right)\right.$$

$$\left. + \frac{W^{-\frac{5}{6}}(\eta)\ W^{\frac{5}{3}}(\eta')\lambda(\eta')}{(\lambda_n - \lambda(\eta'))^2}\right\} \cdot$$

$$\exp\left\{ -a\lambda_n \int_{\eta'}^{\eta} W^{\frac{2}{3}}(\eta)\, W^{-1}(\eta'')\, d\eta''\right\} \cdot\ d\eta' \qquad (28)$$

6. THE ARITHMETICAL PROCEDURE

The ordinary 'section' method of tackling the system (23) would be to confine attention to the first N equations of the set, where N is chosen fairly small, and in these equations to put $\Psi_n = 0$ for $n > N$. Or an 'escalator' process could be used: an iterative process in which the equations are taken in the order

1, 2, 1, 2, 3, 1, 2, 3, 4, 1, ... ;

they are solved respectively for

$$\Psi_1, \Psi_2, \Psi_1, \Psi_2, \Psi_3, \Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_1, \cdots,$$

using at each stage the most recently obtained approximation for the other functions that are involved.

However by using the approximation (27) or (28) for the functions of high order we can substantially improve these processes. For the section method we solve the first N equations after substituting for Ψ_{N+1}, Ψ_{N+2}, ... from (28), say, and for the escalator method we use (28) at each stage for those higher-order functions which have not been approximated to during the escalation. The escalation is stopped when we obtain a Ψ_N which agrees, to an accuracy judged sufficient, with Ψ_N as found from (28); and there is a similar test for adequacy of a chosen N in the section method.

Regarding the system (23) I have not carried through either of these processes to what I should call the bitter end, since I aimed only at about 5 or 10% accuracy in θ.

I have however used escalation for the accurate determination of the constants c_n in the leading approximation

$$\Psi_n \sim k V_o^{\frac{5}{6}} \quad c_n \eta^{\frac{8}{3}}$$

when η is small. Formula (26) gives the estimate

$$c_n \approx \frac{1}{3(\lambda_n - \lambda(o))} = \frac{1}{3(\lambda_n - 1.02)} = c_n' \ .$$

The table below shows these values c_n' for n = 1, 2, ...7, and the values c_n as found by escalation.

n	1	2	3	4	5	6	7
c_n'	.1885	.0960	.0680	.0541	.0456	.0397	.0355
c_n	.2932	.0818	.0729	.0526	.0465	.0395	.0337

7. NOTES

(i) Regarding the determination of θ_1 from the series (15) there is a sting in the tail: the series is only slowly convergent, and to get reasonable accuracy it is necessary to estimate the sum of its tail-end terms, by analytical work based on an approximation such as (26) and the formula (16).

(ii) The object was to find the maximum temperature in the liquid. This came out at about half what it would have been in the case of zero conductivity, and it occurred at about $z = \frac{1}{10}$ h; the approximation whereby (6) was replaced by (8) is therefore justified, to an accuracy of say 10%.

The validity also of the approximation $\theta = 0$ for $z = 0$ was examined a posteriori. From the solution (15) we can calculate the temperature gradient at $z = 0$ in the liquid (as a function of r, t), and thence, knowing the conductivity in the material of the anvil, the temperature gradient in the anvil is found. This gives the boundary condition for solution of the conduction equation in the anvil, whence the surface temperature $\theta = \theta_s$ is found and compared with the approximation $\theta = 0$. In the actual case the surface temperature thus found was about one tenth of the maximum temperature found in the liquid, and the approximation $\theta = 0$ to the surface temperature is thereby justified to about 10%. A better approximation could be found by making a fresh start; we should put $\theta = \theta^1 + \theta^{11}$, when θ^1 is a function chosen, as plausibly as possible, to have the boundary value θ_s, and θ^{11} is a new unknown. The determination of θ^{11} would amount to solving (12) with a modified value for the last term on the right.

DISCUSSION

Dr. I.D. Campbell, D.S.L.

In the light of present day developments in digital computers could Professor Cherry indicate where he would now stop his analysis and transfer to these machines.

Professor T.M. Cherry (In Reply)

One would in general do on these machines what is only talked about here, i.e. solve the differential equations. It must always be remembered that analyses give the relations of parameters in extenso and numerical work cannot provide this. Further this problem is a singular one and such problems always introduce difficulties for the direct numerical approach.

Dr. T. Pearcey, C.S.I.R.O., Sydney.

I think this illustrates something which is going to be important in the future. This is a very practical problem and there are a lot like it with the equivalent complications. It appears to me that the simple equations which we are studying for stability and so on are in practice a very small group. This means that, in the future, we will have to mould our mathematical analyses onto our numerical techniques in the way that Professor Cherry would have tackled his problem now. This required that users of these machines will have to have first-class mathematical training with a good physical background.

Dr. A.S. Douglas, University of Leeds.

I agree with Dr. Pearcey. The only difficulty is at what stage one changes from mathematical physics to obtain the necessary numerical experience. It is usually difficult to obtain any clear cut distinction as to when one changes from analysis to the computing side since it depends so much on the problem, the facilities and the person.

Dr. T. Pearcey, C.S.I.R.O., Sydney.

This is so - but the general trend is still towards solving practical problems which are more complicated and this does seem to imply an increasing standard, or if you like, a more analytical approach before attacking future problems.

Dr. T. Pearcey, C.S.I.R.O., Sydney.

Why do you propose inverting a large matrix for each step instead of using a Richardson type process of this specific type (example given a black board).

Dr. A.S. Douglas, University of Leeds.

I don't know. I haven't tried it but it looks as though it may involve as much work as inverting a matrix.

Dr. T. Pearcey, C.S.I.R.O., Sydney.

I don't think so. It is certainly faster and has the same stability characteristics.

Mr. J.H. Wilkinson, National Physical Laboratory.

I don't see that you are inverting a matrix. You seem to be solving linear equations.

Dr. A.S. Douglas, University of Leeds.

In this case the equation is linear and the step lengths are constants so it is worthwhile.

Professor T.M. Cherry (In Reply)

Regarding the varying steps in r. This suggests from the analytical point of view that a change of variable should be made before you start. It may also suggest changes of dependent variable.

Dr. A.S. Douglas, University of Leeds.

This is true. We did try some variable changes but they did not seem successful at one end or the other.

Dr. T. Pearcey, C.S.I.R.O., Sydney.

Perhaps you need not use an algebraic transformation but you might win out by using a tabulated transformation which suits your purposes at both ends.

Dr. A.S. Douglas, University of Leeds.

It is quite possible that that is the right way to deal with it.